Embedding Cartesian Products of Graphs into de Bruijn Graphs

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Abstract

Given a Cartesian product $G = G_1 \times \ldots \times G_m (m \geq 2)$ of nontrivial connected graphs G_i and the n–dimensional base B de Bruijn graph $D = D_B(n)$, it is investigated whether or not G is a spanning subgraph of D. Special attention is given to graphs $G = G_1 \times \ldots \times G_m$ which are relevant for parallel computing, namely, to Cartesian products of paths (*"m–dimensional grids"*) or cycles (*"m–dimensional tori"*).

For 2–dimensional de Bruijn graphs $D = D_B(2)$, we present a theorem stating that certain structural assumptions on the factors $G_1, \ldots, G_m$ ensure that $G_1 \times \ldots \times G_m$ is a spanning subgraph of D. As corollaries, we obtain results improving previous results of Heydemann, Opatrny, and Sotteau (1994) on embedding grids and tori into de Bruijn graphs. Specifically, we obtain that (i) any grid $G = G_1 \times \ldots \times G_m$ is a spanning subgraph of $D = D_B(2)$ provided that $|G| = |D|$, and (ii) any torus $G = G_1 \times \ldots \times G_m$ is a spanning subgraph of $D = D_B(2)$ provided that $|G| = |D|$ and that the G_i are cycles of even length ≥ 4. Although these results are valid only for de Bruijn graphs of dimension $n = 2$, they do have consequences for the case $n > 2$, too: for even n, we can apply our results to obtain embeddings of grids and tori G into de Bruijn graphs $D_C(n)$ with dilation $n/2$, where the base C is a fixed (typically small) integer ≥ 2, and n is big enough to ensure $|G| \leq |D_C(n)|$.

We demonstrate the practical relevance of these results by means of experimental investigations in the area of parallel image processing. We also contrast our results for $n = 2$ with nonexistence results for $n \geq 3$. We show that $G = G_1 \times \ldots \times G_m$ is never a spanning subgraph of $D = D_B(n)$ when n is greater than three or when n equals three and m is greater than 2. For $n = 3$ and $m = 2$, we show for wide classes of graphs $G = G_1 \times \ldots \times G_m$ that G cannot be a spanning subgraph of D. In particular, these results imply that $D_B(n)$ never contains a torus as a spanning subgraph when n is greater than 2.

Keywords: de Bruijn graphs, Cartesian product, grids, tori, graph embeddings, dilation, processor networks, parallel image processing, massively parallel computers

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1 Introduction

In the context of parallel and distributed computation the problem of embedding one interconnection network into another one is of fundamental importance and has gained considerable attention; see, e.g., the survey article of Monien and Sudborough [10], and the literature mentioned there. De Bruijn graphs are popular communication networks for parallel computers because they feature several nice properties such as fixed node degree and small diameter (for more details cf. [4, 15, 16]).

The problem of studying embeddings of hypercubes, grids and tori into de Bruijn graphs was initiated by Heydemann, Opatrny, and Sotteau [7, 8] who obtained a variety of results on embedding hypercubes and 2-dimensional grids into de Bruijn graphs, while grids of higher dimension and tori were considered only briefly. In [12] results on the embedding of multidimensional tori into de Bruijn graphs were presented. (For definitions, see Section 2.)

In the present paper, given a Cartesian product $G = G_1 \times \ldots \times G_m$ $(m \geq 2)$ of nontrivial connected graphs G_i and the n-dimensional base B de Bruijn graph $D = D_B(n)$, we investigate whether or not there exists a spanning subgraph of D which is isomorphic to G. We apply our results to obtain embeddings of relatively small dilation of common processor networks (as grids, tori, and hypercubes) into de Bruijn graphs of small fixed base. Our results generalize and improve some of the results obtained by Heydemann, Opatrny, and Sotteau [7, 8]. Parts of the present paper were presented in [2].

The paper is organized as follows. In Section 2, we collect some definitions and notational conventions. Thereafter, in Section 3, we present our main results (Theorems 1 and 2) and their corollaries. Experimental results are presented in Section 4. Section 5 contains a somewhat technical part of the proof of Theorem 2 which was postponed in Section 3.

2 Notation

Unless stated otherwise, graphs will be undirected, multiple edges and loops are admitted. Our terminology is standard and (essentially) in accordance with the terminology of [5]; graph-theoretic notions not defined here can be found in [5]. For a graph G, we denote by $V(G)$ and $E(G)$ the set of vertices and edges, respectively. A graph is *simple* if it has neither loops nor multiple edges. The complete graph with n vertices is denoted K_n. The complete bipartite graph with color classes of cardinality n and m, respectively, is denoted $K(n,m)$. A graph is *nontrivial* if it has at least two vertices.

For graphs $G_i (i = 1, \ldots, m)$, the *Cartesian product* $G_1 \times \ldots \times G_m = \bigtimes_{i=1,\ldots,m} G_i$ is the graph with vertex set $V(G_1) \times \ldots \times V(G_m)$ where two distinct vertices $(a_1, \ldots, a_m)$, $(b_1, \ldots, b_m)$ are joined by t (multiple) edges (for $t \geq 1$) if and only if there exists an $i \in \{1, \ldots, m\}$ such that $a_j = b_j$ for all $j \neq i$ and such that there are t edges of G_i joining a_i with b_i;

further, there are $\sum_{i=1}^{m} t_i$ loops at a vertex $(a_1, \ldots, a_m)$, where t_i is the number of loops at a_i in G_i.

Popular interconnection networks in the area of parallel computing are the grids, tori, and hypercubes which, in terms of the Cartesian product, can be defined as follows. A graph G is an *m-dimensional grid* (*m-dimensional torus*) if G is the Cartesian product $G_1 \times \ldots \times G_m$ of m nontrivial paths (cycles). The *m-dimensional binary hypercube* is the graph $G = G_1 \times \ldots \times G_m$ where the G_i are complete graphs on two vertices ($i = 1, \ldots, m$); for $m = 3$, G is called *cube*.

For integers B, $n \geq 2$, the *directed base B de Bruijn graph of dimension n* (denoted $\overrightarrow{D}_B(n)$) has as its vertex set the set of n-tuples $a = \langle a_{n-1}, \ldots, a_0 \rangle$ where the entries a_i are integers with $0 \leq a_i \leq B-1$, and there exists an arc from $a = \langle a_{n-1}, \ldots, a_0 \rangle$ to $b = \langle b_{n-1}, \ldots, b_0 \rangle$ if and only if $b_i = a_{i-1}$ for $i = 1, \ldots, n-1$. The corresponding *undirected base B de Bruijn graph of dimension n* (denoted $D_B(n)$) results from $\overrightarrow{D}_B(n)$ by ignoring the orientation of the arcs. In other words, $D_B(n)$ has the same vertex set as $\overrightarrow{D}_B(n)$ and distinct vertices $a = \langle a_{n-1}, \ldots, a_0 \rangle$ and $b = \langle b_{n-1}, \ldots, b_0 \rangle$ are joined by a single edge if either $b_i = a_{i-1}$ for $i = 1, \ldots, n-1$ or $a_i = b_{i-1}$ for $i = 1, \ldots, n-1$, and by exactly two edges if both of these conditions hold; moreover, there is a loop at $a = \langle a_{n-1}, \ldots, a_0 \rangle$ if $a_0 = a_1 = \ldots = a_{n-1}$.

For graphs G, H an *embedding* $\varphi : G \longrightarrow H$ consists of a pair of mappings φ_1, φ_2 where φ_1 maps each vertex of G onto a vertex of H and φ_2 assigns to every edge ab of G a path $\varphi_2(ab)$ of H which connects $\varphi_1(a)$ with $\varphi_1(b)$. The *dilation* of an embedding is the maximum length of the paths $\varphi_2(ab)$. If φ_1 is injective, then the embedding is said to have *load 1*. If both φ_1 and φ_2 are injective and if each path $\varphi_2(ab)$ consists of an edge connecting $\varphi_1(a)$ with $\varphi_1(b)$, then φ is called a *subgraph embedding*. The ratio $|V(H)|/|V(G)|$ is called *expansion* of an embedding $\varphi : G \rightarrow H$, and sometimes G and H are called the *guestgraph* and the *hostgraph* of φ, respectively.

3 The main results

In this section, we present our results on the question whether or not a given Cartesian product $G = G_1 \times \ldots \times G_m$ ($m \geq 2$) of nontrivial graphs G_i is a spanning subgraph of $D_B(n)$. The proof of our first result (Theorem 1) is based on the following lemma.

Lemma 1 *Let $B = a + b$ for positive integers a, b. Then $K(a, b) \times K(a, b)$ is a spanning subgraph of $D_B(2)$.*

Proof Let A_0, A_1 be disjoint sets with $|A_0| = a$, $|A_1| = b$, and $A_0 \cup A_1 = \{0, \ldots, B-1\}$. For $x, y \in \{0, \ldots, B-1\}$, we write $x \sim y$ if $x, y \in A_i$ for $i = 0$ or 1, and we use

$x \not\sim y$ to indicate that $x \sim y$ does not hold. Let V be the set of pairs (x, y) with $x, y \in \{0, \ldots, B-1\}$. We draw an edge between (x, y), $(x', y') \in V$ if $x = x'$ and $y \not\sim y'$ or $y = y'$ and $x \not\sim x'$. Let E denote the corresponding set of edges. Then the graph $G = (V, E)$ is isomorphic to $K(a, b) \times K(a, b)$. Let $\varphi : V \longrightarrow V(D_B(2))$ be the mapping defined by $\varphi(x, y) = \langle x, y\rangle$ if $x \sim y$ and $\varphi(x, y) = \langle y, x\rangle$, otherwise.

Clearly, φ is bijective. For the purpose of showing that φ defines a subgraph embedding of G into $D_B(2)$, let $(x, y), (x, y') \in V$ with $y \not\sim y'$. By symmetry, we may assume $x \sim y$, $x \not\sim y'$. It follows that $\varphi(x, y) = \langle x, y\rangle$, $\varphi(x, y') = \langle y', x\rangle$, and thus $\varphi(x, y)$ and $\varphi(x, y')$ are neighbors in $D_B(2)$. Clearly, the case $(x, y), (x', y) \in V$, $x \not\sim x'$, can be settled in the same manner, and thus φ defines a subgraph embedding $G \longrightarrow D_B(2)$. ■

For a bipartite graph H, we say that H is of *type* (a, b) if $V(H)$ is the disjoint union of sets A, B with $|A| = a$, $|B| = b$ such that each edge of H joins a vertex of A with a vertex of B.

The following Lemma 2 is an immediate consequence of Lemma 1.

Lemma 2 *For $B \geq 2$, let G be a graph with $|G| = |D_B(2)|$ and let H_1, H_2 be bipartite graphs of the same type. Assume that G is a spanning subgraph of $H_1 \times H_2$. Then G is a spanning subgraph of $D_B(2)$.*

Proof Let (a, b) be the type of H_1 and H_2. Since G is a spanning subgraph of $H_1 \times H_2$, we have $B^2 = |G| = |H_1 \times H_2| = (a+b)^2$. Hence $B = a+b$. Clearly, $H_1 \times H_2$ is a spanning subgraph of $K(a, b) \times K(a, b)$ and thus Lemma 1 can be employed to find that G is a spanning subgraph of $D_B(2)$. ■

In order to present Theorem 1, we need the following definition. Let $G = G_1 \times \ldots \times G_m$ be a Cartesian product of $m \geq 2$ nontrivial graphs and assume that, for some $B \geq 2$, G has the same number of vertices as $D_B(2)$, i.e.,

$$\prod_{i=1}^{m} |G_i| = B^2 \,. \tag{1}$$

For each $i \in \{1, \ldots, m\}$, assume that $q_{i,1}, \ldots, q_{i,r_i} (r_i \geq 1)$ are integers with $q_{i,j} \geq 2\,(j = 1, \ldots, r_i)$ and $|G_i| = q_{i,1} \cdot \ldots \cdot q_{i,r_i}$. Put $M = \Big\{(i, j) : 1 \leq i \leq m,\, 1 \leq j \leq r_i\Big\}$ and, for each $(i, j) \in M$, let $a_{i,j}$, $b_{i,j}$ be positive integers with $q_{i,j} = a_{i,j} + b_{i,j}$. Assume further that M can be partitioned into disjoint subsets M_1, M_2 such that the bipartite graphs

$$H_1 := \bigtimes_{(i,j)\in M_1} K(a_{i,j}, b_{i,j}) \text{ and } H_2 := \bigtimes_{(i,j)\in M_2} K(a_{i,j}, b_{i,j}) \text{ are of the same type.} \tag{2}$$

Then we call the family of triples $(q_{i,j}, a_{i,j}, b_{i,j})$ $(1 \leq i \leq m;\, 1 \leq j \leq r_i)$ a *family of triples associated to* $G_1 \times \ldots \times G_m$. Note that if follows from (1) that there exists at least one family of triples associated to $G_1 \times \ldots \times G_m$: Let $|G_i| = q_{i,1} \cdot \ldots \cdot q_{i,r_i}$ be the factorization of $|G_i|$ into primes and put $a_{i,j} = \lfloor \frac{q_{i,j}}{2} \rfloor$, $b_{i,j} = \lceil \frac{q_{i,j}}{2} \rceil$ $(i = 1, \ldots, m;\ j = 1, \ldots r_i)$. Then (1) implies that, for each prime p, the number of pairs (i, j) with $q_{i,j} = p$ is an even number. Hence M can be partitioned into M_1, M_2 such that (2) holds.

Theorem 1 *Let $G = G_1 \times \ldots \times G_m (m \geq 2)$ be a Cartesian product of nontrivial graphs G_i and assume $|G| = |D_B(2)|$, $B \geq 2$. Let, further, $(q_{i,j}, a_{i,j}, b_{i,j})$ $(1 \leq i \leq m,\ 1 \leq j \leq r_i)$ be a family of triples associated to $G_1 \times \ldots \times G_m$ and assume that, for each $i \in \{1, \ldots, m\}$,*

$$G_i \text{ is a spanning subgraph of } K(a_{i,1}, b_{i,1}) \times \ldots \times K(a_{i,r_i}, b_{i,r_i}). \tag{3}$$

Then G is a spanning subgraph of $D_B(2)$.

Proof For the given family $(q_{i,j},\ a_{i,j},\ b_{i,j})$ $(1 \leq i \leq m,\ 1 \leq j \leq r_i)$ of triples associated to $G_1 \times \ldots \times G_m$, let H_1, H_2 be defined as in (2). Then H_1, H_2 are bipartite graphs of the same type and one obtains from (3) by application of the associative and commutative law for the Cartesian product that G is a spanning subgraph of $H_1 \times H_2$. Hence, by Lemma 2, G is a spanning subgraph of $D_B(2)$. ∎

We now present some corollaries of Theorem 1.

Corollary 1 *Let $G = G_1 \times \ldots \times G_m (m \geq 2)$ be a Cartesian product of even cycles of length ≥ 4 and assume $|G| = |D_B(2)|$ for $B \geq 2$. Then G is a spanning subgraph of $D_B(2)$.*

Proof Let $|G_i| = q_{i,1} \cdot \ldots \cdot q_{i,r_i}$ be the factorization of $|G_i|$ into primes and let $a_{i,j} = \lfloor \frac{q_{i,j}}{2} \rfloor$, $b_{i,j} = \lceil \frac{q_{i,j}}{2} \rceil$. Then, as mentioned above, $(q_{i,j}, a_{i,j}, b_{i,j})$ is a family of triples associated to $G_1 \times \ldots \times G_m$. Note that each $K(a_{i,j}, b_{i,j})$ has a Hamiltonian path. Further, for each $i \in \{1, \ldots, m\}$, there exists a $j \in \{1, \ldots, r_i\}$ such that $K(a_{i,j}, b_{i,j}) \cong K_2$ (since $|G_i|$ is even) and we have $r_i \geq 2$ (since $|G_i| \geq 4$). From these facts one easily concludes that the graph $K(a_{i,1}, b_{i,1}) \times \ldots \times K(a_{i,r_i}, b_{i,r_i})$ has a Hamiltonian cycle $(i = 1, \ldots, m)$, which means that (3) holds. Thus the assertion follows from Theorem 1. ∎

In a similar way one obtains

Corollary 2 *If $G = G_1 \times \ldots \times G_m (m \geq 2)$ is a Cartesian product of nontrivial paths G_i and if $|G| = |D_B(2)|$, $B \geq 2$, then G is a spanning subgraph of $D_B(2)$.*

Note that the Corollaries 1 and 2 cover the cases of bipartite tori, arbitrary grids and, as a special case of Corollary 2, binary hypercubes. The above results improve some of the results of Heydemann et al [7, 8] who obtained the result of Corollary 1 under the additional assumption that the cycle lengths are powers of two (see also [12]); similarly the papers [7, 8] contain the result of Corollary 2 under additional assumptions on the path lengths (covering the hypercube case).

We mention that, in a sense, Corollary 1 is sharp since the Cartesian product $C_3 \times C_3$ is not a spanning subgraph of $D_3(2)$, as can easily be checked. (Here C_t denotes the cycle of length t.) In addition, we found that $C_5 \times C_5$ *is* a spanning subgraph of $D_5(2)$ but do not know whether or not $C_t \times C_t$ is a spanning subgraph of $D_t(2)$ for odd $t \geq 7$. The fact that $C_5 \times C_5$ is a spanning subgraph of $D_5(2)$ can be seen as follows. For $i = 0, 1, 2, 3, 4$ let A_i and B_i be the following 5–cycles contained in $D_5(2)$, where the addition is modulo 5 (cf Figure 1 for an illustration of the construction):

$$\begin{aligned} A_i &= (\langle i, i\rangle, \langle i, 1+i\rangle, \langle 1+i, 4+i\rangle, \langle 4+i, 1+i\rangle, \langle i, 4+i\rangle) \\ B_i &= (\langle i, i\rangle, \langle i, 2+i\rangle, \langle 2+i, 3+i\rangle, \langle 3+i, 2+i\rangle, \langle i, 3+i\rangle). \end{aligned}$$

One finds that the union of cycles A_i, B_i $(i = 0, \cdots, 4)$ is isomorphic to $C_5 \times C_5$.

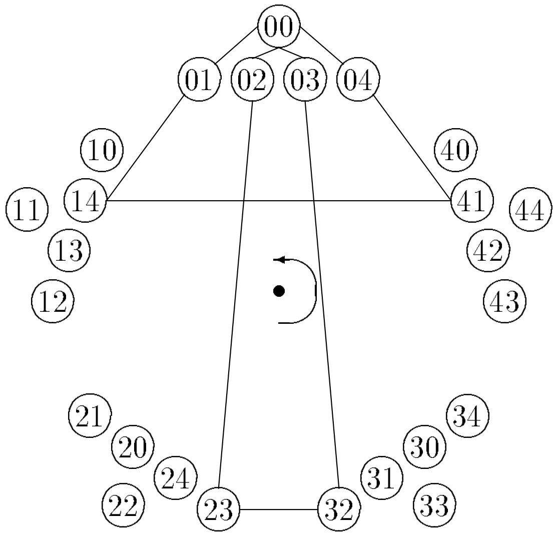

Figure 1: An illustration of $C_5 \times C_5$ as a spanning subgraph of $D_5(2)$.

A Cartesian product $G_1 \times \ldots \times G_m$ of nontrivial graphs G_i is called a *representation* of G if $G = G_1 \times \ldots \times G_m$. For a bipartite graph G, a representation $G_1 \times \ldots \times G_m$ of G is *even* if for each $i \in \{1, \ldots, m\}$, one can choose integers a_i, b_i such that G_i is of type (a_i, b_i) and such that, for every pair (a, b) of positive integers, the number of indices $i \in \{1, \ldots, m\}$ with $(a_i, b_i) = (a, b)$ is even.

Corollary 3 *Let G be a simple bipartite graph with $|G| = |D_B(2)|$, $B \geq 2$, and assume that G possesses an even representation $G_1 \times \ldots \times G_m$. Then G is a spanning subgraph of $D_B(2)$.*

Proof Let $(a_i, b_i)(i = 1, \ldots, m)$ be as in the definition of "even representation". For each $i \in \{1, \ldots, m\}$, put $r_i = 1$, $q_{i,1} = |G_i|$, $a_{i,1} = a_i$, and $b_{i,1} = b_i$. Because $G_1 \times \ldots \times G_m$ is an even representation , it is possible to partition $M = \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq r_i\}$ into disjoint subsets M_1, M_2 such that (2) holds. Hence $(q_{i,j}, a_{i,j}, b_{i,j})$ $(1 \leq i \leq m, 1 \leq j \leq r_i)$ is a family of triples associated to $G_1 \times \ldots \times G_m$. Moreover, condition (3) holds since G is simple. Thus Theorem 1 can be employed to find that G is a spanning subgraph of $D_B(2)$. ∎

Our results on embedding graphs into de Bruijn graphs of dimension 2 can be used to obtain results on embedding graphs into de Bruijn graphs $D = D_C(n)$ of higher dimension, where the base C is an arbitrary fixed integer ≥ 2 and n is an even number which is big enough to ensure that the host graph D has at least as many nodes as the guest graph. The dilation of the so-obtained embeddings is only half the diameter of D which is useful for practical purposes (cf. Section 4). The construction is based on the following well-known proposition on embedding a de Bruijn graph of dimension 2 into a de Bruijn graph of higher dimension (see e.g. [7, 8, 11]).

Proposition 1 *For given integers $B, C \geq 2$, let t be an integer with $t \geq \lceil \log_C B \rceil$. Then there exists an embedding of $D_B(2)$ into $D_C(2t)$ with load 1 and dilation t.*

Proof For each vertex $\langle a, b \rangle$ of $D_B(2)$, $0 \leq a, b < B$, let

$$\varphi_1(\langle a, b \rangle) = \langle a_{t-1}, \ldots, a_0, b_{t-1}, \ldots, b_0 \rangle \in V(D_C(2t))$$

where $a = \sum_{i=0}^{t-1} a_i C^i$ and $b = \sum_{i=0}^{t-1} b_i C^i$ with $a_i, b_i \in \{0, \ldots, C-1\}$, i.e., the a_i and b_i are the coefficients of the base C representation of a and b, respectively. Then, clearly, φ_1 is injective. Further, let $\langle a, b \rangle \langle b, c \rangle$ be any edge of $D_B(2)$ and let $a_i, b_i, c_i \in \{0, \ldots, C-1\}$ for $i = 0, \ldots, t-1$, where the a_i and b_i are defined as before and the c_i are determined by the equation $c = \sum_{i=0}^{t-1} c_i C^i$. Define the vertices $x_0, \ldots, x_t$ of $D_C(2t)$ by

$$\begin{aligned} x_0 &= \langle a_{t-1}, \ldots, a_0, b_{t-1}, \ldots, b_0 \rangle \\ x_i &= \langle a_{t-i-1}, \ldots, a_0, b_{t-1}, \ldots, b_0, c_{t-1}, \ldots, c_{t-i} \rangle (i = 1, \ldots, t-1) \\ x_t &= \langle b_{t-1}, \ldots, b_0, c_{t-1}, \ldots, c_0 \rangle . \end{aligned}$$

Then $x_0 = \varphi_1(\langle a, b \rangle)$, $x_t = \varphi_1(\langle b, c \rangle)$ and x_i is joined to x_{i+1} by an edge of $D_C(2t)$ $(i = 0, \ldots, t-1)$. Thus, by selecting an appropriate subsequence of $x_0, \ldots, x_t$, one obtains a path of length at most t joining $\varphi_1(\langle a, b \rangle)$ with $\varphi_1(\langle b, c \rangle)$. Further, the so-defined path has length exactly t whenever the vertices $x_0, \ldots, x_t$ are pairwise distinct which, for example, is the case for $a = 0$ and $b = 1$. This shows that the constructed embedding has dilation t. ∎

Proposition 1 can be combined with the preceding results of this section in the following way. Let B, C, t be as in Proposition 1 and let G be a spanning subgraph of $D_B(2)$. Then

Proposition 1 provides us with an embedding of G into $D_C(2t)$ with load 1 and dilation t; moreover, if $B = C^t$, then this embedding has expansion 1. Thus any of our previous results ensuring that $G = G_1 \times \ldots \times G_m$ is a spanning subgraph of $D_B(2)$ can be used to obtain an embedding of G into $D_C(2t)$ with load 1, dilation t, and (if $B = C^t$) expansion 1. For example, Corollary 2 yields the following:

> *For integers $C \geq 2$, $t \geq 1$, let G be any grid with C^{2t} vertices. Then G can be embedded into $D_C(2t)$ with load 1, expansion 1, and dilation t.*

In Section 4 we demonstrate how a similar result for tori can be used for applications in the field of parallel image processing.

In the remainder of this section, we present a nonexistence result for de Bruijn graphs $D_B(n)$ with $n \geq 3$ which contrasts the above results for $n = 2$.

Theorem 2 *Let $G = G_1 \times \ldots \times G_m$ be a Cartesian product of $m \geq 2$ nontrivial connected graphs and let $D = D_B(n)$ for $n \geq 3$ and $B \geq 2$. If $m = 2$ and $n = 3$, then assume that G_1, G_2 are loopless, G_1 is not a path and G_2 has no vertices of degree one. Then G is not a spanning subgraph of D.*

We remark that the additional assumption made in Theorem 1 for the case $m = 2$, $n = 3$ cannot be dropped (but, possibly, relaxed): let G_1 and G_2 be paths of length one and three, respectively. Then the grid $G_1 \times G_2$ is a spanning subgraph of $D_2(3)$.

Proof The proof consists of two parts. In part I, Theorem 2 is proved with the exception of the case $m = 2, n = 3$. In part II the case $m = 2, n = 3$ is settled. Since the proof of part II is somewhat technical, it is postponed to Section 5.

Part I: We assume $n \geq 4$ when $m = 2$.
We show how this case can be settled by application of the results proved in [1]. For a direct argument which does not use the more general results of [1], we refer to the original proof of Theorem 2 presented in [2]. Clearly, it suffices to settle the case that the G_i are simple graphs, which is assumed henceforth.

We suppose that G is a subgraph of D and show that this implies $|G| < |D|$. If $n \geq 5$, then it follows from Theorem 2 in [1] that $B \geq |G|/4$. Hence $|D| = B^n > 4B \geq |G|$. If either $m \geq 3$ or $n \geq 4$ and $G = G_1 \times G_2$ with $|G_i| \geq 3$ $(i = 1, 2)$ then it follows form Proposition 1 and Theorem 1 in [1] that $|G| \leq 2|B|$, and thus $|G| \leq 2|B| < B^n = |D|$. Hence it remains to consider the case $n = 4$, $G = G_1 \times K_2$ which we now assume. Then, by Corollary 1 in [1] we have $|G| \leq 2B^3 - B^2 - B$ if $B \geq 3$ and $|G| \leq 12$ if $B = 2$, and thus $|G| < B^4 = |D|$ in either case. ■

As an immediate consequence of Theorem 2, one obtains the following corollary which supplements the above Corollary 1.

Corollary 4 *A torus $G = G_1 \times \ldots \times G_m (m \geq 2)$ cannot be a spanning subgraph of a de Bruijn graph $D_B(n)$ of dimension $n \geq 3$.*

4 Application

This section is addressed to readers particularly interested in practical applications and may be skipped by those mainly concerned with theoretical matters. We describe the practical impact of our results on parallel algorithms in digital image processing; for an introduction to this application field, we refer to [9, 14]. The implementation of an algorithm on a parallel system can be modeled as a graph embedding problem where the host graph is given by the interconnection network, and the guest graph corresponds to the data dependency graph of the algorithm.

For positive integers X, Y, let $\tilde{X} = \{0, \ldots, X-1\}$, $\tilde{Y} = \{0, \ldots, Y-1\}$ and $\tilde{X} \times \tilde{Y} = \{(x, y) : x \in \tilde{X}, y \in \tilde{Y}\}$. In digital image processing, an *image* is defined as a mapping $f = f^{X,Y} : \tilde{X} \times \tilde{Y} \to F$, where the range F is determined by the application under consideration. For the purpose of this paper, F can be assumed to be a field. Moreover, we limit ourselves to quadratic images, i.e. $X = Y =: M$. The M^2 triples $(x, y, f(x, y))$ are called the *pixels* of the image.

A standard image processing application is typically divided into a low level preprocessing stage, a medium level of feature extraction, and a high level of classification. The convolution is a typical example for image preprocessing. For images $f^{M,M}$ and $k^{M,M}$, the *cyclic convolution* of $f^{M,M}$ by $k^{M,M}$, denoted $f^{M,M} * k^{M,M}$, is defined as follows:

$$(f^{M,M} * k^{M,M})(x, y) = \sum_{0 \leq i,j < M} f^{M,M}((x+i) \bmod M, (y+j) \bmod M) \cdot k^{M,M}(i, j).$$

The image $k = k^{M,M}$ is called the *kernel* of the convolution. The *radius* r of the kernel is the smallest value r such that $\min(i, M-i) + \min(j, M-j) > r$ implies $k^{M,M}(i, j) = 0$. Typically, the radius r is small in comparison to the image dimension M. Therefore, the kernel is stored (redundantly) on all nodes in a parallel implementation, whereas the image is distributed among the nodes. A 2-dimensional torus $C_M \times C_M$ would be the natural choice for the interconnection network of a target parallel computer. The embedding of the convolution into $C_M \times C_M$ has dilation r, since the data have to be communicated along paths of length at most r.[1]

[1] More formally, the data dependency graph of the algorithm that computes the convolution is the graph $G = (C_M \times C_M)^r$ ("r-th power of $C_M \times C_M$") and the embedding of G into $C_M \times C_M$ is defined by the identity map on $V(G)$, where each edge of G is mapped on a path of length at most r. In graph theory, the r-th power of a graph H is the graph with the same vertex set as H where two vertices are joined by an edge if and only if their distance in H is at most r.

In contrast to low level image processing, the algorithms associated with the higher levels, as for example Fast Fourier Transform, Viterbi algorithm, or sorting, call for interconnection networks with low diameter. In this context, typical structures are trees, de Bruijn graphs, hypercubes or dense irregular graphs.

Naturally, it is desirable to perform all stages of an algorithm on one machine. Choosing the de Bruijn graph as interconnection network, embeddings of tori or grids into de Bruijn graphs become of significant importance. Theorem 1 provides us with such embeddings, but we have to take into account two practical limitations:

- The number of M^2 pixels of an image exceeds by far the number of nodes in practical interconnection networks.
- De Bruijn graphs of dimension 2 cannot be directly used in practical applications, because of their high node degree.

Both problems can be resolved by means of an appropriate embedding. The former limitation is resolved by mapping the torus $C_M \times C_M$ onto a (smaller) torus $C_N \times C_N$ such that the node (x, y) is mapped onto the node $\left(\left\lfloor \frac{xN}{M} \right\rfloor, \left\lfloor \frac{yN}{M} \right\rfloor\right)$. Then, by combining the above embedding of the convolution into $C_M \times C_M$ with the so-defined embedding $C_M \times C_M \to C_N \times C_N$, one obtains an embedding of the convolution into $C_N \times C_N$ with dilation $\left\lceil \frac{rN}{M} \right\rceil$. The latter limitation calls for embeddings of de Bruijn graphs of dimension 2 into de Bruijn graphs with small base; embeddings of this kind are provided by Proposition 1. In more detail, let $M = 2^m$ and $N = 2^n$ for integers $m \geq n \geq 2$. Then Corollary 1, in conjunction with Proposition 1, provides us with an embedding of $C_N \times C_N$ into $D_2(2n)$ with load 1, expansion 1, and dilation n and thus we obtain an embedding of the convolution algorithm for the image $f^{M,M}$ into $D_2(2n)$ for which the dilation is bounded by $n \left\lceil \frac{rN}{M} \right\rceil$, which is at most half the diameter of $D_2(2n)$ if $r \leq M/N$.

Embeddings of this kind have been integrated into the parallel image processing system PIPS [13]. We present experimental results comparing the embedding derived in this paper with a standard embedding which does not take advantage of Corollary 1 but uses an embedding of $C_N \times C_N$ into $D_N(2)$ with dilation 2 instead. The latter embedding maps every node (x, y) of $C_N \times C_N$ is mapped onto node $\langle x, y \rangle$ of $D_N(2)$. For the evaluation we consider a convolution like filter[2] with varying kernel radius r. The algorithm is applied to a $M \times M = 512 \times 512$ image and computed on a $N^2 = 64$ node base 2 Transputer based de Bruijn network. Figure 2 gives the quotient of the communication latencies (i.e. the maximum time required for the access of data via the interconnection network). It can be seen that the use of Corollary 1 leads to considerably better latencies for $r \leq \frac{M}{N} = \frac{512}{8} = 64$.

One should expect the quotient between the latencies of both embeddings to be approximately 0.5, since the embedding derived from Corollary 1 has half the dilation of the

[2] As we are only interested in the communication latencies, neither the concrete filter algorithm nor the range of the image is of importance.

embedding which does not take advantage of Corollary 1. In our implementation the quotient varies between 0.58 and 0.74 for $r \leq 64$, which is due to software overhead. For $r > 64$, the advantage of the Corollary 1 embedding is diminished since both embeddings have the same (worst case) dilation of $2n$, which is the diameter of $D_2(2n)$.

From the application point of view (in the case $M \geq N^2$) it is also possible to embed the convolution algorithm into the cycle C_{N^2} of length N^2 with dilation $\left\lceil \frac{rN^2}{M} \right\rceil$ which in turn is a subgraph of $D_2(2n)$ (see [6]). For details concerning this embedding, experimental results, and comparison to the above embeddings, we refer to [12, 13].

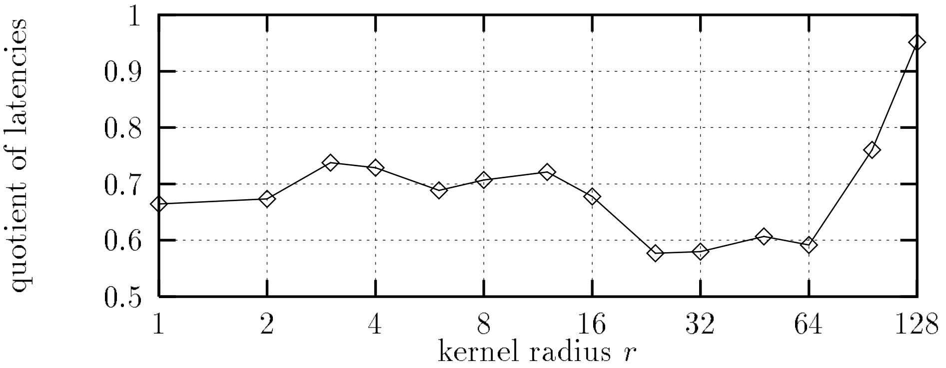

Figure 2: Quotient of the communication latencies of the embedding which takes advantage of Corollary 1 and a standard embedding which does not use Corollary 1 (for a cyclic convolution with varying kernel radius r of a 512×512 image on a 64 node base 2 de Bruijn network).

5 The Second Part of the Proof of Theorem 2

We now present the part of the proof of Theorem 2 which was postponed in Section 3. We need some preparation.

In directed graphs, we write $\overrightarrow{ab}$ to denote an arc from a to b. A *directed trail* (of length t) in a digraph $\overrightarrow{D}$ is a sequence $a_0, \ldots, a_t$ of vertices such that $\overrightarrow{a_i a_{i+1}}$ is an arc of $\overrightarrow{D}$ $(i = 0, \ldots, t-1)$ and such that all arcs $\overrightarrow{a_i a_{i+1}}$ are distinct. For a 4-cycle $C \subseteq D_B(n)$, let $\overrightarrow{C}$ be the corresponding directed subgraph of $\overrightarrow{D}_B(n)$. Then C (as well as $\overrightarrow{C}$) is called a *cycle of type t* if t is the maximum length of a directed trail contained in $\overrightarrow{C}$. Figure 3 displays the different types of 4-cycles. In order to denote the vertices of a 4-cycle of

$\overrightarrow{D}_B(n)$ and to indicate arc directions, we use notations like $a \to b \to c \to d \leftarrow a$ the meaning of which is clear without explanation.

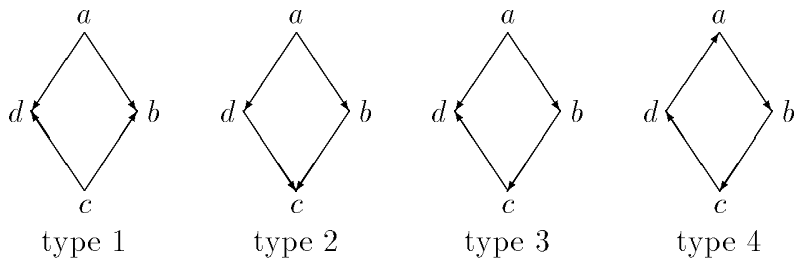

Figure 3: The different types of 4-cycles in $\overrightarrow{D}_B(n)$.

The following Lemmas 3–6 are easy, but useful, consequences of the definitions; cf. [1, 2]. We give proofs of the Lemmas 4 and 6, and leave those of the Lemmas 3 and 5 to the reader.

Lemma 3 *$\overrightarrow{D}_B(n)$ contains no cycle of type 2.* ∎

If $a \to b \to c \to d \leftarrow a$ is a cycle of type 3, then $a \to b \leftarrow c \to d \leftarrow a$ is called *the corresponding cycle of type 1.*

Lemma 4 *If $\overrightarrow{D}_B(n)$ contains a cycle of type 3, then it also contains the corresponding cycle of type 1.*

Proof Let $a \to b \to c \to d \leftarrow a$ be a cycle of type 3 contained in $\overrightarrow{D}_B(n)$ with $a = \langle a_{n-1}, \ldots, a_0 \rangle$. Then $\overrightarrow{ab} \in \overrightarrow{D}_B(n)$ implies $b = \langle a_{n-2}, \ldots, a_0, x \rangle$ for some $x \in \{0, \ldots, B-1\}$, and $\overrightarrow{ad}, \overrightarrow{cd} \in \overrightarrow{D}_B(n)$ implies $c = \langle y, a_{n-2}, \ldots, a_0 \rangle$ for some $y \in \{0, \ldots, B-1\}$. Hence $\overrightarrow{cb} \in \overrightarrow{D}_B(n)$. ∎

Lemma 5 *Let $a \to b \leftarrow c \to d \leftarrow a$ be a cycle of type 1 contained in $\overrightarrow{D}_B(n)$ with $a = \langle a_{n-1}, \ldots, a_0 \rangle$, $b = \langle b_{n-1}, \ldots, b_0 \rangle$, $c = \langle c_{n-1}, \ldots c_0 \rangle$, $d = \langle d_{n-1}, \ldots, d_0 \rangle$. Then $a_{i-1} = c_{i-1} = b_i = d_i (i = 1, \ldots, n-1)$.* ∎

Lemma 6 *Every edge of $\overrightarrow{D}_B(3)$ is contained in at most one cycle of type 4.*

Proof Let $a \to b \to c \to d \to a$ be a cycle of type 4 contained in $\overrightarrow{D}_B(3)$ with $a = \langle a_2, a_1, a_0\rangle$. Then $b = \langle a_1, a_0, a_3\rangle$ for some $a_3 \in \{0, \ldots, B-1\}$. It follows that $c = \langle a_0, a_3, a_2\rangle$ and $d = \langle a_3, a_2, a_1\rangle$, which means that c and d are uniquely determined by a and b. ∎

For a Cartesian product $G_1 \times \ldots \times G_m$, a subgraph H of $G_1 \times \ldots \times G_m$ is called *1-dimensional* if there exists an i such that, for any pair of vertices $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_m)$ of H, $a_j = b_j$ for all $j \neq i$. A 4-cycle $C \subseteq G_1 \times \ldots \times G_m$ which is not 1-dimensional is called *2-dimensional*. For the proof of the next Lemma, we refer to [2, 1].

Lemma 7 *Let $G = G_1 \times \ldots \times G_m$ be the Cartesian product of $m \geq 2$ nontrivial connected graphs and let $\sim$ be an equivalence relation on the vertex set of G. Assume that $a \sim c$ for all vertices a, c which form a pair of opposite vertices on some 2-dimensional 4-cycle of G. Then the partition of $V(G)$ corresponding to $\sim$ consists of at most two classes.*

Before presenting Lemma 8 we need some definitions and notational conventions.

As usual a *walk* in a graph G is a sequence $(v_0, e_1, v_1, \cdots, e_k, v_k)$, whose terms are alternately vertices and edges of G, such that e_i connects the vertices v_{i-1} and v_i $(i = 1, \cdots, k)$. A walk $(v_0, e_1, v_1, \cdots, e_k, v_k)$ is *admissible* if $k \geq 1, v_i \neq v_{i+1} (i = 0, \cdots, k-1)$ and $e_i \neq e_{i+1} (i = 1, \cdots, k-1)$. Note that, in an admissible walk, $v_i = v_{i+2}$ is possible since multiple edges are not excluded.

Let G_1, G_2 be loopless graphs and $G = G_1 \times G_2$. Denote by V_i and E_i the vertex set and edge set of G_i, respectively $(i = 1, 2)$. For every $y \in V_2$ let $V_1^y := \{(x, y) \in V(G) : x \in V_1\}$ and denote by G_1^y the subgraph of G induced by V_1^y. Moreover, let E_1^y denote the edge set of G_1^y and let $\Pi_1^y : G_1 \longrightarrow G_1^y$ be an isomorphism mapping each vertex $x \in V_1$ onto $(x, y) \in V_1^y$ (and each edge connecting x with x' onto an edge connecting (x, y) with (x', y)). Analogously, for each $x \in V_1$, we define G_2^x, V_2^x, E_2^x and Π_2^x. Let $X = (x_0, e_1, x_1, \cdots, e_k, x_k)$ and $Y = (y_0, f_1, y_1, \cdots, f_k, y_k)$ be admissible walks in G_1 and G_2, respectively. For $i = 1, \cdots, k$ let C_i be the 4–cycle formed by the following vertices and edges of G (cf. Figure 4): the vertices of C_i are (x_{i-1}, y_{i-1}), (x_i, y_{i-1}), (x_i, y_i), (x_{i-1}, y_i) and the edges of C_i are $\Pi_1^{y_{i-1}}(e_i)$, $\Pi_2^{x_i}(f_i)$, $\Pi_1^{y_i}(e_i)$, $\Pi_2^{x_{i-1}}(f_i)$. If $k \geq 2$, then we define 4–cycles A_i and B_i for each $i \in \{1, \cdots, k-1\}$ as follows. The vertices of A_i are (x_{i-1}, y_i), (x_i, y_i), (x_i, y_{i+1}), (x_{i-1}, y_{i+1}) and the edges of A_i are $\Pi_1^{y_i}(e_i)$, $\Pi_2^{x_i}(f_{i+1})$, $\Pi_1^{y_{i+1}}(e_i)$, $\Pi_2^{x_{i-1}}(f_{i+1})$; similarly, the vertices of B_i are (x_i, y_{i-1}), (x_{i+1}, y_{i-1}), (x_{i+1}, y_i), (x_i, y_i) and the edges are $\Pi_1^{y_{i-1}}(e_{i+1})$, $\Pi_2^{x_{i+1}}(f_i)$, $\Pi_1^{y_i}(e_{i+1})$, $\Pi_2^{x_i}(f_i)$ (cf. Figure 4). Let $\mathcal{D} = ((A_i)_{i=1}^{k-1}, (B_i)_{i=1}^{k-1}, (C_i)_{i=1}^{k})$ be the triple formed by the so defined sequences of 4–cycles, where $(A_i)_{i=1}^{k-1}$ and $(B_i)_{i=1}^{k-1}$ are empty if $k = 1$. Then we call $\mathcal{D}$ a *diagonal configuration generated by X and Y*. The 4–cycles C_i $(i = 1, \cdots, k)$ are called the *central 4–cycles* of $\mathcal{D}$.

Lemma 8 *Let G_1, G_2 be loopless graphs and $G = G_1 \times G_2$. Let further $X = (x_0, e_1, x_1, \cdots, e_k, x_k)$ and $Y = (y_0, f_1, y_1, \cdots, f_k, y_k)$ be admissible walks of G_1 and G_2, respectively, and let $\mathcal{D} = ((A_i)_{i=1}^{k-1}, (B_i)_{i=1}^{k-1}, (C_i)_{i=1}^{k})$ be a diagonal configuration generated by X*

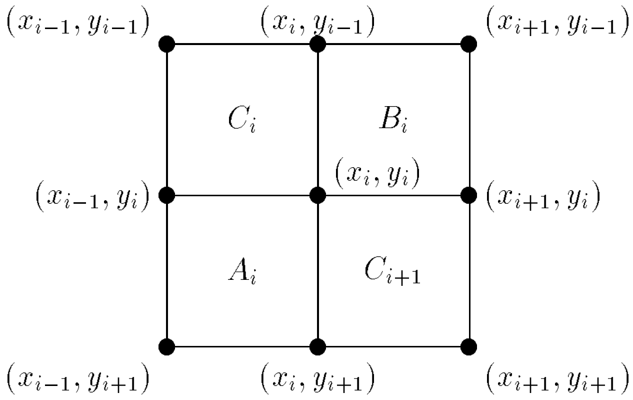

Figure 4: 4-cycles of $G = G_1 \times G_2$ generated by admissible walks X, Y. (Not all vertices shown in the figure are necessarily distinct.)

and Y. Let $D = D_B(3)$ and assume that there exists a subgraph embedding $\varphi : G \longrightarrow D$. Assume further that the 4–cycles $\varphi(C_1), \cdots, \varphi(C_{k-1})$ are of type 4, while $\varphi(C_k)$ is of type $\neq 4$. Let $\varphi((x_0, y_1)) = \langle a_2, a_1, a_0 \rangle$ and $\varphi((x_1, y_0)) = \langle c_2, c_1, c_0 \rangle$. Then $a_1 = c_1$.

Proof For $u, v \in V(D)$ with $u = \langle u_2, u_1, u_0 \rangle$, $v = \langle v_2, v_1, v_0 \rangle$, we write $u \sim v$ if $u_1 = v_1$. By Lemma 6 all 4–cycles $\varphi(A_i), \varphi(B_i)$ are of type $\neq 4$ since $\varphi(A_i)$, as well as $\varphi(B_i)$, is distinct from $\varphi(C_i)$ but shares an edge with $\varphi(C_i)$. Moreover, by assumption, $\varphi(C_k)$ is of type $\neq 4$. Hence we conclude from the Lemmas 3–5 that $\varphi((x_0, y_1)) \sim \varphi((x_1, y_2)) \sim \cdots \sim \varphi((x_{k-1}, y_k)) \sim \varphi((x_k, y_{k-1})) \sim \varphi((x_{k-1}, y_{k-2})) \sim \cdots \sim \varphi((x_1, y_0))$. ■

The degree of a vertex v in a graph G is denoted $d_G(v)$. By $\Delta(G)$ we denote the maximum degree of G.

Proof of Theorem 2, Part II ($m = 2$, $n = 3$) By assumption G_1, G_2 are loopless, G_1 is not a path and G_2 has no vertices of degree one. Suppose that there exists a subgraph embedding $\varphi : G \longrightarrow D$ such that $\varphi(G)$ is a spanning subgraph of D. We choose fixed isomorphisms Π_1^y for $y \in V(G_2)$ and Π_2^x for $x \in V(G_1)$ as described in the paragraph before Lemma 8. Whenever we say that a diagonal configuration $\mathcal{D}$ is generated by some admissible walks of G_1 and G_2, we mean that the construction of $\mathcal{D}$ is based on this fixed choice of the isomorphisms Π_1^y and Π_2^x.

The case $B = 2$ is easily settled. Suppose that $D = D_2(3)$. Note that the graph that results from D by deletion of the loops has minimum degree two. From this, together with the fact that G is loopless, one concludes that G has minimum degree at most two. On the other hand the minimum degree of G_i is at least $i (i = 1, 2)$, which implies that the minimum degree of G is at least three. This contradiction settles the case $B = 2$. Hence let $B \geq 3$.

We first assume that not both G_1 and G_2 are cycles. Note that, under this assumption, it suffices to settle the case that G_1 is not a cycle: this follows from the fact that the condition imposed on G_2 ("no vertices of degree one") is more restrictive than the condition imposed on G_1 ("not a path"). Thus assume that G_1 is not a cycle. Since G_1 is connected and not a path it follows that $\Delta(G_1) \geq 3$. We claim that the following holds.

$$\begin{array}{c}\text{Let } a = \langle a_2, a_1, a_0 \rangle, c = \langle c_2, c_1, c_0 \rangle \text{ be vertices of } D \text{ such that } \varphi^{-1}(a) \\ \text{and } \varphi^{-1}(c) \text{ are opposite vertices of some 2–dimensional 4–cycle } C \text{ of} \\ G. \text{ Then } a_1 = c_1.\end{array} \tag{4}$$

For the proof of (4) let $\varphi^{-1}(a) = (\alpha_1, \alpha_2)$, $\varphi^{-1}(c) = (\gamma_1, \gamma_2)$. Because $\Delta(G_1) \geq 3$ and G_1 is connected, G_1 contains a path $(x_0, e_1, x_1, \cdots, e_\ell, x_\ell)$ with $\ell \geq 1$ such that $d_{G_1}(x_\ell) \geq 3$ and such that either $x_0 = \alpha_1, x_1 = \gamma_1$ or $x_0 = \gamma_1, x_1 = \alpha_1$. We may assume that the former holds since, otherwise, we could interchange the roles of a and c. Because $d_{G_1}(x_\ell) \geq 3$, we can pick distinct edges $e_{\ell+1}, e'_{\ell+1}$ of G_1 which both are distinct from e_ℓ such that $e_{\ell+1}$ and $e'_{\ell+1}$ join x_ℓ with some vertices $x_{\ell+1}$ and $x'_{\ell+1}$, respectively. Then $X_1 = (x_0, e_1, x_1, \cdots, e_\ell, x_\ell, e_{\ell+1}, x_{\ell+1})$ and $X_2 = (x_0, e_1, x_1, \cdots, e_\ell, x_\ell, e'_{\ell+1}, x'_{\ell+1})$ are admissible walks of G_1. Since G_2 has no vertex of degree one, we can find an admissible walk $Y' = (y_0,\ f_1,\ y_1,\ \cdots,\ f_\ell,\ y_\ell,\ f_{\ell+1},\ y_{\ell+1})$ of G_2 with $y_0 = \gamma_2$ and $y_1 = \alpha_2$. Let $\mathcal{D}_1 = ((A_i^1)_{i=1}^{\ell}, (B_i^1)_{i=1}^{\ell}, (C_i^1)_{i=1}^{\ell+1})$ be the diagonal configuration generated by X_1 and Y', and $\mathcal{D}_2 = ((A_i^2)_{i=1}^{\ell}, (B_i^2)_{i=1}^{\ell}, (C_i^2)_{i=1}^{\ell+1})$ be the one generated by X_2 and Y'. Because $C_{\ell+1}^1$ and $C_{\ell+1}^2$ are distinct 4–cycles of G sharing an edge, it follows from Lemma 6 that $\varphi(C_{\ell+1}^1)$ or $\varphi(C_{\ell+1}^2)$ is of type $\neq 4$, and we may assume that the former holds. For the least integer k such that $\varphi(C_k^1)$ is of type $\neq 4$, let

$$\mathcal{D} = ((A_i^1)_{i=1}^{k-1}, (B_i^1)_{i=1}^{k-1}, (C_i^1)_{i=1}^{k})$$

be the diagonal configuration that results from $\mathcal{D}_1$ by deleting the last $(\ell+1-k)$ 4–cycles from each of the sequences of $\mathcal{D}_1$. Observe that $\varphi((x_0, y_1)) = \varphi((\alpha_1, \alpha_2)) = \langle a_2, a_1, a_0 \rangle$ and $\varphi((x_1, y_0)) = \varphi((\gamma_1, \gamma_2)) = \langle c_2, c_1, c_0 \rangle$. Hence $a_1 = c_1$ by Lemma 8, which proves (4).

From (4), together with Lemma 7, one finds that $|G| \leq 2B^2$. Hence $B^3 = |D| = |G| \leq 2B^2$ which implies $B = 2$, contradicting the fact that $B \geq 3$. Hence it remains to consider the case that both G_1 and G_2 are cycles, which we now assume.

Now we can finish the proof by application of [1], Theorem 4, as follows. Clearly $\varphi : G \to D$ cannot be a trivial embedding because this would imply $2B \geq |G| = |D| = B^3$. (For the definition of the term "trivial embedding", see [1].) Hence φ is nontrivial. If both G_1, G_2 are simple, i.e., cycles of length greater than two, then Theorem 4 of [1] implies that G_1 and G_2 are cycles of the same length ℓ for some $\ell \equiv 2 \pmod 4$ and $B \geq \max\left\{\frac{\ell}{2}, 5\right\}$. Hence $B^3 = |D| = |G| = \ell^2$; further, $B \geq \frac{\ell}{2}$ if $\ell \geq 10$ and $B \geq 5$ if $\ell = 6$. Consequently, $B^3 \leq 4B^2$ if $\ell \geq 10$ and $125 \leq B^3 = \ell^2$ if $\ell = 6$. Clearly, the latter is impossible and the former implies $B = 3$ or 4, which contradicts the fact that $B^3 = \ell^2$ for $\ell \equiv 2 \pmod 4$.

If G_1 or G_2 is a cycle of length two, then $\varphi(G) \subseteq D_B(3)$ would imply that $D_B(3)$ contains four distinct vertices x, y, z, w such that x and y as well as z and w are connected by a pair of double-edges and such that x and z are neighbors in $D_B(3)$. But this implies $x = \langle a, b, a\rangle$, $y = \langle b, a, b\rangle$, $z = \langle c, d, c\rangle$, $w = \langle d, c, d\rangle$, and $\{a, b\} = \{c, d\}$, in contradiction to the assumption that x, y, z, w are pairwise distinct. ∎

Remark. In the above proof, the case that G is a Cartesian product of two cycles is settled by making use of [1], Theorem 4. We mention that a more direct proof can be based on Lemma 8 without employing the results of [1] and, in fact, the original proof found by the present authors was of this type. The interested reader is referred to [3].

6 Conclusion

This paper presents conditions for the existence (or non-existence) of *optimal* embeddings, i.e., embeddings with unit dilation, load, and expansion, of Cartesian products of graphs into de Bruijn graphs.

In particular, we prove that Cartesian products of graphs cannot be optimally embedded into de Bruijn graphs of dimension greater than 3. In the case of de Bruijn graphs of dimension 3 this non-existence result is extended to large classes of Cartesian product graphs including the case of more than two factors and the case of 2-dimensional tori graphs, which are of particular interest for practical applications.

For de Bruijn graphs of dimension 2, optimal embeddings of Cartesian products of graphs do exist under some additional conditions. The proof of this result is constructive and covers such important cases as multidimensional grids, binary hypercubes, and tori with even side length.

Some questions concerning optimal embeddings into de Bruijn graphs of dimension 2 and 3 have been left open and we expect that, for these cases, different methods have to be employed in order to obtain complete answers.

For practical applications one is mainly interested in embeddings into de Bruijn graphs of small base, e.g., of base 2. Therefore the obtained embeddings have to be combined with embeddings transforming base and dimension of de Bruijn graphs. In total, we obtain embeddings of Cartesian products of graphs into de Bruijn graphs of fixed base with dilation at most half the diameter of the de Bruijn graph.

Experimental results have shown that the efficiency of parallel implementations can be improved considerably by means of the obtained embeddings. Especially this is the case for algorithms having communication demands that can be described by tori and grids at one stage and by de Bruijn graphs at another stage. These kind of mixed communication demands are typical for applications in the area of digital image processing.

References

[1] T. Andreae, M. Hintz, M. Nölle, G. Schreiber, G.W. Schuster, and H Seng. Cartesian products of graphs as subgraphs of de Bruijn graphs of dimension at least three. *Discrete Applied Mathematics*. Accepted for publication.

[2] T. Andreae, M. Nölle, and G. Schreiber. Cartesian Products of Graphs as Spanning Subgraphs of de Bruijn Graphs. In *Proceedings of the 20. Workshop on Graph-Theoretic Concepts in Computer Science (WG94)*, pages 140–150. Lecture Notes in Computer Science 903, Springer, June 1994.

[3] T. Andreae, M. Nölle, and G. Schreiber. Embedding Cartesian Products of Graphs into de Bruijn Graphs. Internal Report 3/95, Technische Informatik I, TU-HH, February 1995.

[4] J.C. Bermond and C. Peyrat. De Bruijn and Kautz networks: a competitor for the hypercube? In F. Andre and J. P. Verjus, editors, *Proceedings of the 1st European Workshop on Hypercube and Distributed Computers*, pages 279–293, Amsterdam, 1989. North-Holland.

[5] J.A. Bondy and U.S.R. Murty. *Graph Theory with Applications*. Macmillian, London, 1977.

[6] N.G. de Bruijn. A Combinatorial Problem. *Koninklijke Nederlandsche Akademie van Wetenschappen Proc.*, Vol. 49, pages 758–764, 1946.

[7] M.C. Heydemann, J. Opatrny, and D. Sotteau. Embeddings of hypercubes and grids into de Bruijn graphs. Rapport de Recherche 723, LRI, December 1991.

[8] M.C. Heydemann, J. Opatrny, and D. Sotteau. Embeddings of hypercubes and grids into de Bruijn graphs. *Journal of Parallel and Distributed Computing*, Vol. 23, pages 104–111, 1994.

[9] B. Jähne. *Digital image processing*. Springer, Berlin, 3rd edition, 1995.

[10] B. Monien and H. Sudborough. Embedding one Interconnection Network in another. *Computing Supplementum*, Vol. 7, pages 257–282, 1990.

[11] M. Nölle. *Konzepte zur Entwicklung paralleler Algorithmen der digitalen Bildverarbeitung*. PhD thesis, Technische Universität Hamburg-Harburg, December 1994. Fortschritt-Berichte, VDI-Reihe 10, Nr. 410, VDI-Verlag, 1996.

[12] M. Nölle and G. Schreiber. Einbettung von Gitter–Algorithmen in de Bruijn–Graphen. *3. PASA Workshop Bonn, in Mitteilungen–Gesellschaft für Informatik e.V.*, pages 12–19, April 1993.

[13] M. Nölle, G. Schreiber, and H. Schulz-Mirbach. PIPS–a general purpose Parallel Image Processing System. In W.G. Kropatsch and H. Bischof, editors, *16. DAGM - Symposium "Mustererkennung"*, pages 609–623, Wien, September 1994. Reihe Informatik XPress, TU-Wien.

[14] W. K. Pratt. *Digital image processing.* Wiley, New York, 2nd edition, 1991.

[15] M.R. Samatham and D.K. Pradhan. The de Bruijn Multiprocessor Network: A Versatile Parallel Processing and Sorting Network for VLSI. *IEEE Trans. on Computers*, Vol. 38, pages 567–581, April 1989.

[16] H.S. Stone. Parallel Processing with the Perfect Shuffle. *IEEE Trans. on Computers*, Vol. C-20, pages 153–161, February 1971.